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Various Continuities of Metric Projections in $C_0(T, X)$

WU LI*

Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang, People's Republic of China

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We give a unified approach to lower semicontinuity and almost lower semicontinuity of metric projections P_G in $C_0(T, X)$, where X is a strictly convex Banach space. We obtain a characterization theorem on pointwise lower semicontinuity of P_G and prove that P_G has a continuous selection if and only if P_G is almost lower semicontinuous. © 1989 Academic Press, Inc.

1. INTRODUCTION

Recently, the problems concerning various continuities of metric projections in the Banach space $C_0(T)$ of real-valued continuous functions have been deeply investigated [3, 4, 6, 7, 8, 9, 13, 17–20, 23]. There were some efforts to generalize the results in $C_0(T)$ to $C_0(T, X)$, where X is a strictly convex Banach space [5, 21]. In this paper, we give a new approach to perturb a given function in $C_0(T, X)$. This provides a unified way to study lower semicontinuity, almost lower semicontinuity, and continuous selections of metric projections P_G in $C_0(T, X)$. Some analogous theorems as those in $C_0(T)$ are obtained or reproved in a new way.

In Section 2, we give a theorem (Theorem 2.5) about perturbation of a given function; In Section 3, by using the perturbation theorem, we show that P_G has a continuous selection if and only if P_G is almost lower semicontinuous (Theorem 3.3). In Section 4, we establish a criterion about pointwise lower semicontinuity of P_G (Theorem 4.1) and reprove a characterization theorem about lower semicontinuity of P_G (Corollary 4.3).

Now we introduce some notations. Let T be a locally compact Hausdorff space and X a strictly convex Banach space. $C_0(T, X)$ will denote the

* Present address: Department of Mathematics, Pennsylvania State University, University Park, PA 16802, U.S.A.

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Banach space of continuous mappings f from T to X which vanish at infinity, i.e., the set $\{t \in T: \|f(t)\|_X \geqslant \varepsilon\}$ is compact for each $\varepsilon > 0$. The norm of f in $C_0(T, X)$ is defined as

$$\|f\| = \sup\{\|f(t)\|_X : t \in T\}.$$

For $G \subset C_0(T, X)$, the metric projection P_G from $C_0(T, X)$ to G is

$$P_G(f) = \{g \in G: \|f - g\| = d(f, G)\}, \qquad f \in C_0(T, X),$$

where

$$d(f, G) = \inf\{\|f - p\| : p \in G\}.$$

In this paper, G will always denote a finite-dimensional subspace of $C_0(T, X)$ and the following notations will be used throughout:

$S(X) :=$ the unit sphere of X,
$E(F) := \{t \in T: \|f(t)\|_X = \|f\| \text{ for all } f \text{ in } F\}$,
$Z(F) := \{t \in T: f(t) = 0 \text{ for all } f \text{ in } F\}$,
$\text{card}(A) :=$ the cardinal number of A,
$G(A) := \{g \in G: A \subset Z(g)\}$,
$G|_A := \{g|_A : g \in G\}$,

where A denotes a subset of T and F denotes a subset of $C_0(T, X)$.

2. Perturbation of a Given Function

LEMMA 2.1 [10]. *Suppose that $f \in C_0(T, X) \backslash G$ and $g \in G$. Then $g \in P_G(f)$ if and only if there exist $\{t_i\}_1^m \subset T$ and $\{\varphi_i\}_1^m \subset X^* \backslash \{0\}$ such that*

(1) $\sum_{i=1}^m \varphi_i(f(t_i) - g(t_i)) = \|f - g\| \cdot \sum_{i=1}^m \|\varphi_i\|$;
(2) $\sum_{i=1}^m \varphi_i(p(t_i)) = 0$, *for* $p \in G$.

Remark. The characterization condition given in [10] is slightly different from conditions (1) and (2). But it is easy to see that they are equivalent.

Now we are going to establish several technical lemmas for the proof of the perturbation theorem (Theorem 2.5).

LEMMA 2.2. *For every $f \in C_0(T, X)$, there is a $g^* \in P_G(f)$ such that $E(f - g^*) = E(f - P_G(f)) \subset \{t \in T: g^*(t) = g(t) \text{ for all } g \in P_G(f)\}$.*

Proof. Let g^* be in the relative interior of $P_G(f)$. Then for any $g \in P_G(f)$, there is an $\varepsilon > 0$ such that

$$g^* + \lambda(g^* - g) \in P_G(f), \qquad \text{for} \quad |\lambda| \leqslant \varepsilon.$$

Now for any $t \in E(f - g^*)$, we have

$$\begin{aligned} &\|f(t) - g^*(t) - \lambda(g^*(t) - g(t))\| \\ &\quad \leqslant \|f - g^* - \lambda(g^* - g)\| = d(f, G) = \|f - g^*\| \\ &\quad = \|f(t) - g^*(t)\|, \qquad \text{for} \quad |\lambda| \leqslant \varepsilon, \end{aligned}$$

which implies

$$g^*(t) - g(t) = 0, \qquad t \in E(f - g^*), \quad g \in P_G(f),$$

since X is strictly convex. Thus,

$$\begin{aligned} \|f(t) - g(t)\| &= \|f(t) - g^*(t)\| = \|f - g^*\| \\ &= \|f - g\|, \qquad t \in E(f - g^*), \quad g \in P_G(f), \end{aligned}$$

i.e.,

$$E(f - g^*) = E(f - P_G(f)). \quad \blacksquare$$

LEMMA 2.3. *Suppose that $d(f, G) = 1$ and $E(f - P_G(f)) \setminus \operatorname{int} Z(G) \neq \varnothing$. Then there exist $g^* \in P_G(f)$, $A_k \subset T$ with $\operatorname{card}(A_k) < \infty$, and mappings ψ_k from A_k to $S(X)$ such that*

$$(1) \quad \lim_{k \to \infty} \max\{\|\psi_k(t) - (f(t) - g^*(t))\| : t \in A_k\} = 0; \tag{2.1}$$

$$(2) \quad \dim G|_{\bigcup_{j=1}^{\infty} A_j} = \dim G|_{A_k} \geqslant 1, \qquad \textit{for} \quad k \geqslant 1; \tag{2.2}$$

$$(3) \quad P_{G|_{A_k}}(\psi_k) = \{0\}, \qquad \textit{for} \quad k \geqslant 1. \tag{2.3}$$

Proof. Set $f_k = f|_{T_k}$ and $G_k = G|_{T_k}$ where

$$T_k = \{t \in T : \sup\{\|g(t)\| : g \in G \text{ with } \|g\| = 1\} \geqslant 1/k\}.$$

Then

$$\bigcup_{k=1}^{\infty} T_k = T \setminus Z(G). \tag{2.4}$$

Let $g_k \in G$ such that

$$g_k|_{T_k} \in P_{G_k}(f_k). \tag{2.5}$$

By Lemma 2.1, there exist $B_k = \{t_{i,k}\}_1^{m_k} \subset T_k$ and $\{\varphi_{i,k}\}_1^{m_k} \subset X^* \backslash \{0\}$ such that

$$\sum_{i=1}^{m_k} \varphi_{i,k}(f(t_{i,k}) - g_k(t_{i,k})) = d(f_k, G_k) \cdot \sum_{i=1}^{m_k} \|\varphi_{i,k}\|; \tag{2.6}$$

$$\sum_{i=1}^{m_k} \varphi_{i,k}(p(t_{i,k})) = 0, \qquad \text{for} \quad p \in G. \tag{2.7}$$

Since G is finite-dimensional, by selecting a subsequence, we may assume that

$$\dim G|_{\bigcup_{j=1}^\infty B_j} = \dim G|_{\bigcup_{j=k}^\infty B_j}, \qquad k \geqslant 1; \tag{2.8}$$

$$\lim_{k \to \infty} g_k = g^* \in G. \tag{2.9}$$

By (2.4), (2.5), (2.9), and $E(f - P_G(f)) \backslash \operatorname{int} Z(G) \neq \varnothing$, it is not difficult to verify that

$$\lim_{k \to \infty} d(f_k, G_k) = d(f, G) = 1, \tag{2.10}$$

$$g^* \in P_G(f). \tag{2.11}$$

Meanwhile, (2.8) implies that there exist $0 = j_1 < j_2 < \cdots$ such that

$$\dim G|_{\bigcup_{j=1}^\infty A_j} = \dim G|_{\bigcup_{j=1}^\infty B_j} = \dim G|_{A_k} \geqslant 1, \qquad k \geqslant 1. \tag{2.12}$$

where

$$A_k = \bigcup_{i=j_k+1}^{j_{k+1}} B_i, \qquad k \geqslant 1.$$

Define

$$\psi_k(t) = \begin{cases} (f(t) - g_{j_k+1}(t))/d(f_{j_k+1}, G_{j_k+1}), & t \in B_{j_k+1}, \\ (f(t) - g_i(t))/d(f_i, G_i), & t \in B_i \backslash \bigcup_{s=j_k+1}^{i} B_s, \quad j_k + 1 < i \leqslant j_{k+1}. \end{cases}$$

By (2.6) we know that ψ_k are mappings from A_k to $S(X)$. Since X is strictly convex, it is not difficult to show that (2.6) and (2.7) imply

$$P_{G|_{B_j}}((f - g_j)|_{B_j}) = \{0\}, \qquad j \geqslant 1. \tag{2.13}$$

By using induction and (2.13), we can easily show that

$$P_{G|_{A_k}}(\psi_k) = \{0\}, \qquad k \geqslant 1. \tag{2.14}$$

It follows from (2.10) and (2.9) that

$$\lim_{k\to\infty} \max\{\|\psi_k(t)-(f(t)-g^*(t))\|: t\in A_k\}=0. \tag{2.15}$$

By (2.11), (2.12), (2.14), and (2,15), we can see that A_k, ψ_k, and g^* satisfy (2.1)–(2.3). ∎

LEMMA 2.4. *If $f\in C_0(T, X)$ with $d(f, G)=1$, then there exist $g^*\in P_G(f)$, $A_k\subset T$ with* $\operatorname{card}(A_k)<\infty$, *and mappings ψ_k from A_k to $S(X)$ such that*

$$(1)\quad \lim_{k\to\infty} \max\{\|\psi_k(t)-(f(t)-g^*(t))\|: t\in A_k\}=0; \tag{2.16}$$

$$(2)\quad P_{G|_{A_k}}(\psi_k)=\{0\}, \qquad k\geqslant 1; \tag{2.17}$$

$$(3)\quad E(f-g^*)\subset \operatorname{int} Z(G(A_k)), \qquad k\geqslant 1. \tag{2.18}$$

Proof. If $E(f-P_G(f))\subset \operatorname{int} Z(G)$, by Lemma 2.2, choose $g^*\in P_G(f)$ such that $E(f-g^*)=E(f-P_G(f))$. Let $t_0\in E(f-P_G(f))$, $\psi_k(t_0)=f(t_0)-g^*(t_0)$, $A_k=\{t_0\}$. Then (2.16)–(2.18) hold. So, without loss of generality, we may assume $E(f-P_G(f))\backslash \operatorname{int} Z(G)\neq\varnothing$. We proceed with the proof by induction on $\dim G$.

If $\dim G=1$, then Lemma 2.4 follows from Lemma 2.3, since $G(A_k)=\{0\}$ for all A_k in Lemma 2.3. Suppose that the conclusion of Lemma 2.4 is true if $\dim G\leqslant s$. Now assume $\dim G=s+1$. By Lemma 2.3, there exist $g_1\in P_G(f)$, $A_{1,k}\subset T$ with $\operatorname{card}(A_{1,k})<\infty$, and mappings $\psi_{1,k}$ from $A_{1,k}$ to $S(X)$ such that

$$\lim_{k\to\infty} \max\{\|\psi_{1,k}(t)-(f(t)-g_1(t))\|: t\in A_{1,k}\}=0; \tag{2.19}$$

$$\dim G|_{\bigcup_{j=1}^{\infty}A_{1,j}}=\dim G|_{A_{1,k}}\geqslant 1, \qquad k\geqslant 1; \tag{2.20}$$

$$P_{G|_{A_{1,k}}}(\psi_{1,k})=\{0\}, \qquad k\geqslant 1. \tag{2.21}$$

Set

$$G^*=\{g\in G: A_{1,k}\subset Z(g) \text{ for all } k\geqslant 1\},$$

$$f^*=f-g_1.$$

Then it is easy to see that $d(f^*, G^*)=d(f, G)$. By (2.20), we get $\dim G^*\leqslant s$ and

$$G^*=\{g\in G: A_{1,k}\subset Z(g)\}=: G(A_{1,k}), \qquad k\geqslant 1. \tag{2.22}$$

By the inductive hypothesis, there exist $g_2 \in P_{G^*}(f^*)$, $A_{2,k} \subset T$ with $\text{card}(A_{2,k}) < \infty$, and mappings $\psi_{2,k}$ from $A_{2,k}$ to $S(X)$ such that

$$\lim_{k \to \infty} \max\{\|\psi_{2,k}(t) - (f^*(t) - g_2(t))\| : t \in A_{2,k}\} = 0; \tag{2.23}$$

$$P_{G^*|_{A_{2,k}}}(\psi_{2,k}) = \{0\}, \qquad k \geqslant 1; \tag{2.24}$$

$$E(f^* - g_2) \subset \text{int } Z(G^*(A_{2,k}), \qquad k \geqslant 1. \tag{2.25}$$

Set

$$A_k = A_{1,k} \cup A_{2,k},$$
$$g^* = g_1 + g_2,$$
$$\psi_k(t) = \begin{cases} \psi_{1,k}(t), & t \in A_{1,k}, \\ \psi_{2,k}(t), & t \in A_{2,k} \backslash A_{1,k}. \end{cases}$$

Obviously, $g^* \in P_G(f)$, $\text{card}(A_k) < \infty$, and ψ_k are mappings from A_k to $S(X)$. Since $g_2 \in G^*$, $A_{1,k} \subset Z(g_2)$ for all $k \geqslant 1$. By (2.19) and (2.23), we obtain

$$\begin{aligned} &\lim_{k \to \infty} \max\{\|\psi_k(t) - (f(t) - g^*(t))\| : t \in A_k\} \\ &\quad \leqslant \lim_{k \to \infty} (\max\{\|\psi_{1,k}(t) - (f(t) - g_1(t))\| : t \in A_{1,k}\} \\ &\qquad + \max\{\|\psi_{2,k}(t) - (f^*(t) - g_2(t))\| : t \in A_{2,k}\}) = 0. \end{aligned} \tag{2.26}$$

Equations (2.22) and (2.25) imply

$$\begin{aligned} E(f - g^*) = E(f^* - g_2) &\subset \text{int } Z(G^*(A_{2,k})) \\ &= \text{int } Z(G(A_k)), \qquad k \geqslant 1. \end{aligned} \tag{2.27}$$

Now suppose $g \in P_{G|_{A_k}}(\psi_k)$. Then

$$\begin{aligned} &\max\{\|\psi_k(t) - g(t)\| : t \in A_k\} \\ &\quad \leqslant \max\{\|\psi_k(t)\| : t \in A_k\} = 1. \end{aligned} \tag{2.28}$$

Equations (2.21) and (2.28) imply $g(t) = 0$ for $t \in A_{1,k}$. By (2.22), $g \in G^*|_{A_k}$. Similarly, it follows from (2.24) and (2.28) that $g(t) = 0$ for $t \in A_{2,k}$. Thus $g \equiv 0$, i.e.,

$$P_{G|_{A_k}}(\psi_k) = \{0\}. \tag{2.29}$$

Equations (2.26), (2.27), and (2.29) show that A_k, ψ_k, and g^* satisfy (2.16), (2.17), and (2.18). This completes the proof of this lemma. ∎

THEOREM 2.5. *If* $f \in C_0(T, X)\backslash G$, *then there exist* $g^* \in P_G(f)$ *and an open set* $V \supset E(f-g^*)$ *such that for any* $\varepsilon > 0$, *there is an* f_ε *in* $C_0(T, X)$ *satisfying*

$$(1) \quad \|f - f_\varepsilon\| < \varepsilon; \tag{2.30}$$

$$(2) \quad P_G(f_\varepsilon) = \{g \in P_G(f) \colon V \subset Z(g - g^*)\}. \tag{2.31}$$

Proof. Without loss of generality, we may assume $d(f, G) = 1$. By Lemma 2.4, there exist $g^* \in P_G(f)$, $A_k \subset T$ with $\operatorname{card}(A_k) < \infty$, and mappings ψ_k from A_k to $S(X)$ such that (2.16)–(2.18) hold.

Since dim G is finite, there is an open set $V \supset E(f-g^*)$ such that for any $g \in G$ with $E(f-g^*) \subset \operatorname{int} Z(g)$, there holds $V \subset Z(g)$. Set

$$\delta = 1 - \max\{\|f(t) - g^*(t)\| \colon t \in T\backslash V\} > 0.$$

It follows from (2.16) that for some $N > 0$,

$$\max\{\|\psi_k(t) - (f(t) - g^*(t))\| \colon t \in A_k\} < \delta, \qquad k \geqslant N. \tag{2.32}$$

Since $\|\psi_k(t)\| = 1$ for $t \in A_k$, (2.32) implies

$$A_k \subset V, \qquad k \geqslant N.$$

Suppose $A_k = \{t_{i,k} \colon 1 \leqslant i \leqslant m_k\}$. Then there are open sets $V_{i,k}$ such that for $1 \leqslant i \leqslant m_k$, $k \geqslant N$,

$$V_{i,k} \cap V_{j,k} = \varnothing, \qquad 1 \leqslant j \leqslant m_k, \quad i \neq j; \tag{2.33}$$

$$t_{i,k} \in V_{i,k} \subset V; \tag{2.34}$$

$$\|(f(t_{i,k}) - g^*(t_{i,k})) - (f(t) - g^*(t))\| < 1/k, \qquad t \in V_{i,k}. \tag{2.35}$$

Let $b_{i,k} \in C_0(T, \mathbb{R})$ such that

$$b_{i,k}(t_{i,k}) = 1;$$
$$0 \leqslant b_{i,k}(t) \leqslant 1, \qquad t \in T;$$
$$b_{i,k}(t) = 0, \qquad t \in T\backslash V_{i,k}.$$

Define

$$f_k(t) = \sum_{i=1}^{m_k} \psi_k(t_{i,k}) \cdot b_{i,k}(t) + (f(t) - g^*(t)) \cdot \left(1 - \sum_{i=1}^{m_k} b_{i,k}(t)\right) + g^*(t).$$

Since $\|f-g^*\| = d(f, G) = 1$, it is easy to check that

$$\begin{aligned}
&\|f_k(t) - g^*(t)\| \\
&\leqslant \sum_{i=1}^{m_k} \|\psi_k(t_{i,k})\| \cdot b_{i,k}(t) \\
&\quad + \|f(t) - g^*(t)\| \cdot \left(1 - \sum_{i=1}^{m_k} b_{i,k}(t)\right) \\
&\leqslant \sum_{i=1}^{m_k} b_{i,k}(t) + \left(1 - \sum_{i=1}^{m_k} b_{i,k}(t)\right) = 1.
\end{aligned}$$

Now, for any $g \in P_G(f_k)$, we have

$$\begin{aligned}
1 &\geqslant \|f_k - g^*\| \geqslant \|f_k - g\| \\
&\geqslant \max\{\|f_k(t_{i,k}) - g(t_{i,k})\| : 1 \leqslant i \leqslant m_k\} \\
&= \max\{\|\psi_k(t_{i,k}) - (g(t_{i,k}) - g^*(t_{i,k}))\| : 1 \leqslant i \leqslant m_k\} \\
&\geqslant d(\psi_k, G|_{A_k}) = 1,
\end{aligned}$$

which implies

$$d(f_k, G) = 1; \tag{2.36}$$

$$(g - g^*)|_{A_k} \in P_{G|_{A_k}}(\psi_k). \tag{2.37}$$

By (2.18) and (2.37), we obtain $A_k \subset Z(g - g^*)$, i.e., $g - g^* \in G(A_k)$. It follows from (2.17) that

$$E(f - g^*) \subset \operatorname{int} Z(G(A_k)) \subset \operatorname{int} Z(g - g^*),$$

which implies

$$V \subset Z(g - g^*).$$

By (2.34) and the definition of $b_{i,k}$, $f_k(t) = f(t)$ for $t \in T \backslash V$, $k \geqslant N$. Thus,

$$\begin{aligned}
\|f(t) - g(t)\| &= \|f(t) - g^*(t)\| \leqslant 1, \qquad t \in V; \\
\|f(t) - g(t)\| &= \|f_k(t) - g(t)\| \leqslant 1, \qquad t \in T \backslash V.
\end{aligned}$$

The above two inequalities imply $g \in P_G(f)$. Hence,

$$P_G(f_k) \subset \{g \in P_G(f) : V \subset Z(g - g^*)\}. \tag{2.38}$$

On the other hand, for any $g \in P_G(f)$ with $V \subset Z(g-g^*)$, we have

$$\|f_k(t)-g(t)\| = \|f_k(t)-g^*(t)\| \leqslant 1, \qquad t \in V, \quad k \geqslant N;$$
$$\|f_k(t)-g(t)\| = \|f(t)-g(t)\| \leqslant 1, \qquad t \in T \backslash V, \quad k \geqslant N,$$

which imply $g \in P_G(f_k)$ for $k \geqslant N$. Thus,

$$P_G(f_k) \supset \{g \in P_G(f) \colon V \subset Z(g-g^*)\}, \qquad k \geqslant N. \tag{2.39}$$

By (2.35) and the definition of f_k, we can derive

$$\begin{aligned}
&\|f(t)-f_k(t)\| \\
&= \left\| \sum_{i=1}^{m_k} b_{i,k}(t) \cdot (\psi_k(t_{i,k}) - (f(t)-g^*(t))) \right\| \\
&\leqslant \max\{\sup\{\|\psi_k(t_{i,k}) - (f(t)-g^*(t))\| \colon t \in V_{i,k}\} : 1 \leqslant i \leqslant m_k\} \\
&\leqslant \max\{\sup\{\|(f(t_{i,k}) - g^*(t_{i,k})) \\
&\quad - (f(t)-g^*(t))\| \colon t \in V_{i,k}\} : 1 \leqslant i \leqslant m_K\} \\
&\quad + \max\{\|\psi_k(t_{i,k}) - (f(t_{i,k}) - g^*(t_{i,k}))\| \colon 1 \leqslant i \leqslant m_k\} \\
&\leqslant 1/k + \max\{\|\psi_k(t_{i,k}) - (f(t_{i,k}) - g^*(t_{i,k}))\| \colon 1 \leqslant i \leqslant m_k\}.
\end{aligned} \tag{2.40}$$

It follows from (2.40) and (2.16) that

$$\lim_{k \to \infty} \|f - f_k\| = 0.$$

Now, for any $\varepsilon > 0$, choose $n \geqslant N$ such that

$$\|f - f_n\| < \varepsilon. \tag{2.41}$$

Then, by (2.38), (2.39), and (2.41), $f_\varepsilon = f_n$ satisfies (2.30) and (2.31). ∎

Remark. Theorem 2.5 provides a new approach to perturb a given function which is quite different from the methods used before (cf. [4, 6, 13, 17, 18]). We will see its efficacy in the following sections.

3. Almost Lower Semicontinuity and Continuous Selection

Recall [12] that P_G is almost lower semicontinuous (alsc) at f if, for any $\varepsilon > 0$, there is an open neighborhood V of f in $C_0(T, X)$ such that

$$\bigcap_{h \in V} \{g \in G \colon d(g, P_G(h)) < \varepsilon\} \neq \varnothing.$$

P_G is said to be alsc if P_G is alsc at every $f \in C_0(T, X)$. Following the notation used by Brown [6], we define

$$P_G^*(f) = \{g \in P_G(f) \colon \lim_{n \to \infty} f_n = f \quad \text{implies} \quad \lim_{n \to \infty} d(g, P_G(f_n)) = 0\}.$$

By [11, Lemma 3.1], we have the following conclusion:

LEMMA 3.1. *P_G is alsc at f if and only if $P_G^*(f) \neq \varnothing$.*

P_G is said to have a continuous selection if there exists a continuous mapping Q from $C_0(T, X)$ to G such that $Q(f) \in P_G(f)$ for each $f \in C_0(T, X)$. The concept of almost lower semicontinuity, introduced by Deutsch and Kenderov [12] for the study of set-valued mappings, is closely related to the existence of continuous selections of set-valued mappings. It follows from a general result of Deutsch and Kenderov [12] that if P_G has a continuous selection, then P_G is alsc. Fischer [14] and Li [18], independently, proved that if G is a finite-dimensional subspace of $C_0(T, \mathbb{R})$ ($=: C_0(T)$) and P_G is alsc, then P_G has a continuous selection. That gave a positive answer to a problem proposed by Deutsch in [7]. Now, by using Theorem 2.5, we can generalize Fischer's and Li's results:

THEOREM 3.2. *If P_G is alsc, then P_G has a continuous selection.*

From Theorem 3.2 and Deutsch and Kenderov's result mentioned above follows the following theorem:

THEOREM 3.3. *P_G has a continuous selection if and only if P_G is almost lower semicontinuous.*

We will prove Theorem 3.2 by showing that P_G^* is lsc if P_G is alsc. First, we need some technical lemmas.

LEMMA 3.4. *If there exist $g^* \in P_G(f)$ and an open set $V \supset E(f - g^*)$ such that*

$$\lim_{\varepsilon \to 0+} \sup_{\|f-h\| < \varepsilon} \{ \inf_{p \in P_G(h)} (\sup_{t \in V} \|g^*(t) - p(t)\|)\} = 0, \tag{3.1}$$

then

$$P_G^*(f) \supset \{g \in P_G(f) \colon V \subset Z(g - g^*)\}.$$

Proof. Assume that Lemma 3.4 fails to be true. Then for some p in

$P_G(f)$ with $V \subset Z(p-g^*)$, $p \notin P_G^*(f)$, i.e., there are f_n and $\delta > 0$ such that for $n \geqslant 1$,

$$\|f - f_n\| < 1/n,$$
$$d(p, P_G(f_n)) \geqslant \delta.$$

By (3.1), there exist $g_n \in P_G(f_n)$ such that

$$\lim_{n \to \infty} \sup\{\|g_n(t) - g^*(t)\| : t \in V\} = 0.$$

By selecting a subsequence, we may assume

$$\lim_{n \to \infty} g_n = p^* \in P_G(f).$$

Then

$$V \subset Z(g^* - p^*) \cap Z(p - g^*) \subset Z(p - p^*).$$

Set

$$\begin{aligned} p_{\lambda,n} &= g_n + (1-\lambda)\cdot(p - p^*) + \lambda\cdot(g^* - p^*), \\ p_\lambda &= (1-\lambda)\cdot p + \lambda\cdot g^*, \\ \eta &= d(f, G) - \max\{\|f(t) - g^*(t)\| : t \in T\backslash V\} > 0. \end{aligned}$$

Then, for $0 < \lambda < 1$,

$$\|f_n(t) - p_{\lambda,n}(t)\| = \|f_n(t) - g_n(t)\| \leqslant d(f_n, G), \qquad t \in V;$$

and for $t \in T\backslash V$,

$$\begin{aligned} &\|f_n(t) - p_{\lambda,n}(t)\| \\ &\quad\leqslant \|f_n(t) - f(t)\| + \|f(t) - p_\lambda(t)\| + \|g_n(t) - p^*(t)\| \\ &\quad\leqslant 1/n + (1-\lambda)\cdot\|f - p\| + \lambda\cdot\|f(t) - g^*(t)\| + \|g_n - p^*\| \\ &\quad\leqslant 1/n + (1-\lambda)\cdot d(f, G) + \lambda\cdot(d(f, G) - \eta) + \|g_n - p^*\| \\ &\quad= d(f_n, G) - \lambda\cdot\eta + 1/n + (d(f, G) - d(f_n, G)) + \|g_n - p^*\|. \end{aligned}$$

Thus for $0 < \lambda < 1$, there are $N(\lambda) > 0$ such that

$$\|f_n - p_{\lambda,n}\| \leqslant d(f_n, G), \qquad n \geqslant N(\lambda),$$

i.e., $p_{\lambda,n} \in P_G(f_n)$ for $n \geqslant N(\lambda)$. Hence, for $0 < \lambda < 1$,

$$\begin{aligned} 0 < \delta &\leqslant \liminf_{n \to \infty} d(p, P_G(f_n)) \\ &\leqslant \|p - p_\lambda\| + \liminf_{n \to \infty} d(p_\lambda, P_G(f_n)) \\ &\leqslant \|p - p_\lambda\| + \liminf_{n \to \infty} \|p_\lambda - p_{\lambda,n}\| \\ &= \|p - p_\lambda\| = \lambda \cdot \|p - g^*\|, \end{aligned}$$

which is impossible. The contradiction completes the proof of this lemma. ∎

LEMMA 3.5. *If P_G is alsc at $f \in C_0(T, X)$, then there $g^* \in P_G(f)$ and an open set $V \supset E(f - g^*)$ such that for any $\varepsilon > 0$, there is f_ε in $C_0(T, X)$ satisfying*

$$(1) \quad \|f - f_\varepsilon\| < \varepsilon; \tag{3.2}$$

$$(2) \quad P_G(f_\varepsilon) = \{g \in P_G(f) \colon V \subset Z(g - g^*)\} = P_G^*(f). \tag{3.3}$$

Proof. The conclusion is trivial if $f \in G$. So we may assume $f \notin G$. By Theorem 2.5, there exist $g^* \in P_G(f)$ and an open set $V \supset E(f - g^*)$ such that

$$\|f - f_\varepsilon\| < \varepsilon; \tag{3.4}$$

$$P_G(f_\varepsilon) = \{g \in P_G(f) \colon V \subset Z(g - g^*)\}. \tag{3.5}$$

Since P_G is alsc at f, by Lemma 3.1, $P_G^*(f) \neq \varnothing$. It follows from (3.4) and (3.5) that

$$\varnothing \neq P_G^*(f) \subset \{g \in P_G(f) \colon V \subset Z(g - g^*)\}, \tag{3.6}$$

which implies

$$\lim_{\varepsilon \to 0+} \sup_{\|f - h\| < \varepsilon} \inf_{g \in P_G(h)} \sup_{t \in V} \|g(t) - g^*(t)\| = 0.$$

By Lemma 3.4, we obtain

$$P_G^*(f) \supset \{g \in P_G(f) \colon V \subset Z(g - g^*)\}. \tag{3.7}$$

(3.4)–(3.7) imply (3.2) and (3.3). ∎

Suppose that Q is a mapping from $C_0(T, X)$ to 2^G; i.e., $Q(f)$ is a subset of G for each $f \in C_0(T, X)$. Recall that Q is lower semicontinuous (lsc) at

f if, for each subset W of $C_0(T, X)$ with $Q(f) \cap W \neq \varnothing$, there is an open neighborhood V of f in $C_0(T, X)$ such that $Q(h) \cap W \neq \varnothing$ for each $h \in V$. Equivalently, P_G is lsc at f if and only if

$$\begin{aligned} Q(f) &= Q^*(f) \\ &:= \{g \in Q(f): \lim_{n \to \infty} f_n = f \quad \text{implies} \quad \lim_{n \to \infty} d(g, P_G(f_n)) = 0\}. \end{aligned}$$

Q is said to be lsc if Q is lsc at every f in $C_0(T, X)$.

THEOREM 3.6. *If P_G is alsc, then P_G^* is lsc.*

Proof. Fix $f \in C_0(T, X)$ and $\varepsilon > 0$. For $h \in C_0(T, X)$ with $\|f - h\| < \varepsilon$, by Lemma 3.5, there is $h_\varepsilon \in C_0(T, X)$ such that

$$\begin{aligned} &\|h - h_\varepsilon\| < \varepsilon - \|f - h\|; \\ &P_G(h_\varepsilon) = P_G^*(h). \end{aligned} \tag{3.8}$$

Equation (3.8) implies $\|f - h_\varepsilon\| < \varepsilon$. Thus, for any $g \in P_G^*(f)$,

$$d(g, P_G^*(h)) = d(g, P_G(h_\varepsilon)) \leqslant \sup_{\|f - f^*\| < \varepsilon} d(g, P_G(f^*)),$$

i.e.,

$$\sup_{\|f - h\| < \varepsilon} d(g, P_G^*(h)) \leqslant \sup_{\|f - f^*\| < \varepsilon} d(g, P_G(f^*)), \; g \in P_G^*(f) \tag{3.9}$$

By the definition of $P_G^*(f)$ and (3.9), we obtain

$$\lim_{\varepsilon \to 0+} \sup_{\|f - h\| < \varepsilon} d(g, P_G^*(h)) = 0, \qquad g \in P_G^*(f),$$

which implies that P_G^* is lsc at f. Hence, P_G^* is lsc. ∎

Proof of Theorem 3.2. It follows from theorem 3.6 and the Michael selection theorem [22] that P_G^* has a continuous selection Q. Since $Q(f) \in P_G^*(f) \subset P_G(f)$, Q is a continuous selection for P_G. ∎

Remark. In more general case, Beer studied the lower semicontinuity of P_G^*. He showed that if P_G contracts to P_G^* uniformly in a certain sense, then P_G^* is lsc [2]. In [14], Fischer proved results similar to those in Theorem 3.4 in the semi-infinite optimization setting.

4. Characterization of Pointwise Lower Semicontinuity

THEOREM 4.1. *P_G is lsc at $f \in C_0(T, X)$ if and only if*

$$E(f - P_G(f)) \subset \operatorname{int}\{t \in T\colon t \in Z(g-p) \text{ for all } g, p \in P_G(f)\} =: V. \tag{4.1}$$

Proof.

NECESSITY. *Since P_G is lsc at f, $P_G(f) = P_G^*(f)$. By Lemma* 3.5, *there exist $g^* \in P_G(f)$ and an open set $W \supset E(f-g^*)$ such that*

$$P_G(f) = P_G^*(f) = \{g \in P_G(f)\colon W \subset Z(g-g^*)\}. \tag{4.2}$$

Equation (4.2) *implies* (4.1).

SUFFICIENCY. *By Lemma* 2.2, *there is $g^* \in P_G(f)$ such that $E(f-g^*) = E(f - P_G(f))$. Then the open set $V \supset E(f-g^*)$. We claim*

$$\lim_{\varepsilon \to 0+} \sup_{\|f-h\| < \varepsilon} \inf_{g \in P_G(h)} \sup_{t \in V} \|g(t) - g^*(t)\| = 0. \tag{4.3}$$

In fact, if (4.3) fails to be true, then for some $\delta > 0$ there exist f_n in $C_0(T, X)$ such that $\|f - f_n\| < 1/n$ and

$$\sup_{t \in V} \|g(t) - g^*(t)\| \geqslant \delta, \; g \in P_G(f_n), \qquad n \geqslant 1. \tag{4.4}$$

By choosing a subsequence, we may assume that for some $g_n \in P_G(f_n)$,

$$\lim_{n \to \infty} g_n = g \in P_G(f).$$

Since $V \subset Z(g-g^*)$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \sup_{t \in V} \|g_n(t) - g(t)\| \\ &\quad = \lim_{n\to\infty} \sup_{t \in V} \|g_n(t) - g^*(t)\| \\ &\quad \leqslant \lim_{n\to\infty} \|g_n - g^*\| = 0, \end{aligned}$$

which contradicts (4.4). Thus, (4.3) holds. It follows from (4.3) and Lemma 3.4 that

$$P_G^*(f) \supset \{g \in P_G(f)\colon V \subset Z(g-g^*)\} = P_G(f),$$

which implies that P_G is lsc at f. ∎

Remark. For $X = \mathbb{R}$ (the real line), this theorem was announced in [4] as an unpublished theorem of Blatter. Theorem 4.1 can also be derived from the proof of Theorem 3 and Theorem 9 in [5]. But our proof is new and is a bonus of the new perturbation method. Theorem 4.1 was announced and used in [21] to prove an intrinsic characterization condition of the lower semicontinuity of P_G.

COROLLARY 4.2. *P_G is lsc if and only if* (4.1) *holds for every f in $C_0(T, X)$.*

COROLLARY 4.3 (Brosowski and Wegmann [5]). *P_G is lsc if and only if the set $\{t \in T\colon t \in Z(g-p)$ for all $p, g \in P_G(f)\}$ is open for every f in $C_0(T, X)$.*

Proof. We only sketch the proof. Write

$$M(h) := \{t \in T\colon t \in Z(g-p) \quad \text{for all} \quad p, g \in P_G(f)\}.$$

By Lemma 2.2, $E(f - P_G(f)) \subset M(f)$. The sufficiency follows immediately from Corollary 4.2. Now suppose that P_G is lsc. Fix $f \in C_0(T, X)$. If $M(f)$ is not open, let $t^* \in bdM(f)$. Then we can modify f near t^* to construct a new function f^* in $C_0(T, X)$ such that (cf. [5] for the details)

$$t^* \in E(f^* - P_G(f^*)) \backslash \text{int } M(f^*).$$

which contradicts Corollary 4.2. ∎

COROLLARY 4.4. *For any $f \in C_0(T, X)$ and $\varepsilon > 0$, there is an f_ε in $C_0(T, X)$ such that*

(1) $\|f - f_\varepsilon\| < \varepsilon$;

(2) *P_G is lsc at f_ε;*

(3) $P_G(f_\varepsilon) \subset P_G(f)$.

Proof. By Theorem 2.5, there exist $g^* \in P_G(f)$ and an open set $V \supset E(f - g^*)$ such that for any $\varepsilon > 0$, there is an f_ε in $C_0(T, X)$ satisfying

$$\|f - f_\varepsilon\| < \varepsilon; \tag{4.5}$$

$$P_G(f_\varepsilon) = \{g \in P_G(f)\colon V \subset Z(g - g^*)\}. \tag{4.6}$$

Let

$$\eta = d(f, G) - \max\{\|f(t) - g^*(t)\|\colon t \in T \backslash V\}.$$

Then, for $t \in T \setminus V$.

$$\begin{aligned} &\|f_\varepsilon(t) - g^*(t)\| \\ &\quad \leqslant \|f(t) - g^*(t)\| + \|f - f_\varepsilon\| \\ &\quad \leqslant d(f, G) - \eta + \varepsilon \\ &\quad \leqslant d(f_\varepsilon, G) - \eta + \varepsilon + (d(f, G) - d(f_\varepsilon, G)). \end{aligned}$$

Since $d(\cdot, G)$ is a continuous function on $C_0(T, X)$, there is a $\delta > 0$ such that

$$\|f_\varepsilon(t) - g^*(t)\| < d(f_\varepsilon, G), \qquad t \in T \setminus V, \quad 0 < \varepsilon < \delta,$$

which implies

$$E(f_\varepsilon - P_G(f_\varepsilon)) \subset E(f_\varepsilon - g^*) \subset V, \qquad 0 < \varepsilon < \delta.$$

Hence, by (4.6),

$$E(f_\varepsilon - P_G(f_\varepsilon)) \subset V \subset \operatorname{int}\{t \in T\colon t \in Z(g - p) \text{ for all } g, p \in P_G(f_\varepsilon)\}.$$

It follows from Theorem 4.1 that P_G is lsc at f_ε for each $0 < \varepsilon < \delta$. This fact together with (4.5) and (4.6) shows that f_ε satisfies (4.2)–(4.4) for $0 < \varepsilon < \delta$. ∎

The next result follows immediately from Corollary 4.4.

COROLLARY 4.5. *P_G is always lsc on a dense subset of $C_0(T, X)$.*

Remark. Professor Deutsch kindly informed me that Corollary 4.5 also follows from a general result of Fort [15] (or Kenderov [16]). From that general result we can obtain a stronger version of Corollary 4.5, which says that P_G is always lsc on a dense G_δ subset of $C_0(T, X)$.

In [4], Blatter and Schumaker studied the uniqueness of continuous selections of P_G. In the remaining part of this section, we will show the relation between the uniqueness of continuous selections for P_G and the almost Chebyshev property of G.

Recall [4] that Q is called a submapping of P_G if $Q(f) \subset P_G(f)$ for every f in $C_0(T, X)$. Q is called a maximal lsc submapping of P_G if Q is lsc and for any lsc submapping S of P_G, S is a submapping of Q.

COROLLARY 4.6. *Suppose that P_G has a continuous selection. Then $P_G^*(f) = \{S(f)$: S is a continuous selection for $P_G\}$, i.e., P_G^* is the maximal lsc submapping of P_G. Moreover, P_G has a unique continuous selection if and* *only if the lower semicontinuity of P_G at f always implies that $P_G(f)$ is a singleton.*

Proof. Let $Q(f) = \{S(f)$: S is a continuous selection of $P_G\}$. Then Q is the maximal lsc submapping of P_G [4].

By theorem 3.6, P_G^* is lsc. So, P_G^* is a submapping of Q. Since $S(f) \in P_G^*(f)$ for any $f \in C_0(T, X)$ and any continuous selection S of P_G, Q is also a submapping of P_G^*. Thus $P_G^* = Q$ is the maximal lsc submapping of P_G.

Obviously, P_G has a unique continuous selection if and only if $P_G^*(f)$ is a singleton for each $f \in C_0(T, X)$.

If P_G has a unique continuous selection and P_G is lsc at f, then $P_G(f) = P_G^*(f)$ is a singleton.

Now assume that the lower semicontinuity of P_G at f always implies that $P_G(f)$ is a singleton. Fix $f \in C_0(T, X)$ and $g_1, g_2 \in P_G^*(f)$. For any $\varepsilon > 0$, by Corollary 4.5, there is an f_ε in $C_0(T, X)$ such that P_G is lsc at f_ε and $\|f - f_\varepsilon\| < \varepsilon$. Since $P_G(f_\varepsilon)$ is a singleton, we have

$$\begin{aligned}\|g_1 - g_2\| &\leqslant \lim_{\varepsilon \to 0+} (\|g_1 - P_G(f_\varepsilon)\| + \|g_2 - P_G(f_\varepsilon)\|) \\ &\leqslant \lim_{\varepsilon \to 0+} (d(g_1, P_G(f_\varepsilon)) + d(g_2, P_G(f_\varepsilon))) = 0.\end{aligned}$$

Hence, $P_G^*(f)$ is a singleton, i.e., P_G has a unique continuous selection. ∎

We say that G is a Z-subspace of $C_0(T, X)$ if no $g \in G \backslash \{0\}$ vanishes on an open subset of T. If G is a Z-subspace of $C_0(T, X)$, by Theorem 4.1, the lower semicontinuity of P_G at f always implies that $P_G(f)$ is a singleton. So, from Corollary 4.6 follows Corollary 4.7.

COROLLARY 4.7. *Suppose that G is a Z-subspace of $C_0(T, X)$. Then P_G has at most one continuous selection.*

Remark. If T is compact and $X = \mathbb{R}$, Corollary 4.7 reduces to a result of Brown [6].

Now assume that T is a compact metric space and $C_0(T, \mathbb{R}) =: C(T)$. Recall [1] that G is an almost Chebyshev subspace of $C(T)$ if $P_G(f)$ is a singleton for each $f \in C(T)$, except a set of first category in $C(T)$. Bartelt and Schmidt [1] proved that G is an almost Chebyshev subspace of $C(T)$ if and only if the lower semicontinuity of P_G at f always implies that $P_G(f)$ is a singleton. By this result and Corollary 4.6, we have the following corollary.

COROLLARY 4.8. *Suppose that T is a compact metric space, G is a finite-dimensional subspace of $C(T)$, and P_G has a continuous selection. Then P_G has a unique continuous selection if and only if G is an almost Chebyshev subspace of $C(T)$.*

G is an almost Chebyshev subspace of $C[a, b]$ if and only if G is a Z-subspace of $C[a, b]$ [1]. Thus from Corollary 4.8 follows Corollary 4.9.

COROLLARY 4.9 (Blatter and Schumaker [4]). *Suppose that G is a finite-dimensional subspace of $C[a, b]$ and P_G has a continuous selection. Then P_G has a unique continuous selection if and only if G is a Z-subspace of $C[a, b]$.*

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